$$P_2\Phi_1(b)+\Phi_2(b)=-\frac{bdG}{(1-\nu)K(k)}-pb^2\left[1-\frac{E(k)}{K(k)}\right] \qquad (2.8)$$

Equations (2.7) and (2.8) imply that when $\lambda\to\infty$, the value of N_b coincides with that obtained in the problem on splitting an elastic plane with a wedge of finite width [8]. Moreover, when $\lambda\to\infty$, the expression (2.4), with (2.2) taken into account, defining $\gamma(x)$ coincides with the analogous expression obtained in [8].

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CONVERGENCE OF THE PROBLEM OF LIMIT EQUILIBRIUM

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The theory of limit equilibrium of a perfectly plastic body [1] usually deals with the systems possessing a finite number of degrees of freedom. The results obtained for the models of finite dimensions can be extended to the problems of limit equilibrium of solid bodies, using the methods of mathematical programing.

In the present paper we consider a perfectly plastic body of finite volume V with surface S. A load proportional to the parameter P is applied at a part of the surface denoted by S_σ. Conditions of zero displacements $u_i=0$ $(i=1,2,3)$ (u denotes the rate of displacement vector) are given at the remainder S_u of the surface. The stress field must satisfy the equations of equilibrium and the following boundary conditions on S_σ

$$\sigma_{ij,j}=0, \qquad \sigma_{ij}\nu_j-Pk_i=0 \qquad (1)$$

Conditions of plasticity are assumed to have the form of a convex operator $f(\sigma)\leqslant 0$. Under these conditions the problem of limit equilibrium in static formulation consists in determination of $P^*=\sup P(\sigma)$. This corresponds to the generalized Lagrange's functional [2]

$$L(P, \sigma, \mathbf{u}, \lambda) = P + \int\limits_V [u_i \sigma_{ij,j} - \lambda f(\sigma)]\, dv + \int\limits_S u_i (\sigma_{ij}\nu_j - Pk_i)\, ds \tag{2}$$

The generalized saddle point

$$L^* = \sup_{P,\sigma} \inf_{u,\lambda} L(P, \sigma, \mathbf{u}, \lambda) \qquad (\lambda \geqslant 0) \tag{3}$$

corresponds to the limiting static load.

The transposed extremal procedures which also define a saddle point

$$L^{**} = \inf_{u,\lambda} \sup_{P,\sigma} L(P, \sigma, \mathbf{u}, \lambda) \qquad (\lambda \geqslant 0) \tag{4}$$

corresponds to the limiting kinematic load.

The saddle points ($L^* = L^{**} = P^*$) will coincide when the conditions of plasticity have the form of the plastic potential, i.e. when there exists an associated law of plastic flow $\varepsilon_{ij} = \lambda df/d\sigma_{ij}$ where $\lambda \geqslant 0$ denote the plasticity parameters and ε_{ij} denote the components of the rate of plastic deformation tensor.

The integrals appearing in (2) should naturally be regarded as linear functionals in a Hilbert space. Thus we can say that $\mathbf{u}$ is the element of the space of displacements H_u $\sigma \in H_\sigma$, where H_σ is a stress space operator-conjugated with H_u. Let us select from H_u a basis $\{\varphi_1, \varphi_2 ...\}$ such that $\varphi_\alpha = 0$ ($\alpha = 1,2,...$) on S_u. Expansion of the rates of displacement over this basis satisfies the condition of zero displacement on S_u

$$u_i = \sum_\alpha x_i^\alpha \varphi_\alpha \qquad (i = 1, 2, 3) \tag{5}$$

The equation of the amount of work performed by the internal and external forces is

$$\int\limits_V \sigma_{ij}\varepsilon_{ij} dv - P \int\limits_S k_i u_i ds = 0$$

and substitution of (5) into the latter yields

$$\sum_\alpha x_i^\alpha \left(\int\limits_V \sigma_{ij}\varphi_{\alpha,j} dv - P \int\limits_S k_i \varphi_\alpha ds \right) = 0 \tag{6}$$

The fact that the stress field under which (6) holds for any $\mathbf{u} \in H_u$ is feasible from the point of view of statics, can be proved as follows. Integrating (6) by parts we obtain

$$\sum_\alpha x_i^\alpha \left(\int\limits_S \varphi_\alpha (\sigma_{ij}\nu_j - Pk_i)\, ds - \int\limits_V \varphi_\alpha \sigma_{ij,j} dv \right) = 0 \tag{7}$$

If the stress field is feasible from the point of view of statics (i.e. Eqs. (1) hold), then (7) becomes an identity for any x_i^α. Consequently (6) must hold for any x_i^α, and splits into separate equations

$$\int\limits_V \sigma_{ij}\varphi_{\alpha,j} dv - P \int\limits_S k_i \varphi_\alpha ds = 0 \qquad (i = 1, 2, 3;\ \alpha = 1, 2, \ldots) \tag{8}$$

The above set of equations is equivalent to (1) and can be obtained, when the discontinuities in the stress field are orthogonal to the discontinuities in the rate of deformation field. We can also obtain (8) from the condition $\partial L / \partial \mathbf{u} = 0$ for the functional (2), which corresponds to the operation inf in (3).

Suppose that a basis $\{\psi_1, \psi_2 ...\}$ exists in H_σ, for which the conditions $\psi_\beta \nu_i = c_\beta k_i$ ($i = 1, 2,3;\ \beta = 1, 2, ...$) hold on S_σ. Here c_β are any numbers including zero, restricted by the inequality $|c_1| + |c_2| + ... < \infty$. Expansion of the stresses over such a basis satisfies the following conditions on S_σ for some value of the parameter P :

$$\sigma_{ij} = \sum_\beta \zeta_{ij}{}^\beta \psi_\beta, \qquad \sigma_{ij}\nu_j = k_i \sum_\beta c_\beta \tag{9}$$

Here $\zeta_{ij}{}^\beta$ are tensors symmetrical in their subscripts. Substitution of (9) into (8) yields the following set of linear equations:

$$\sum_{\beta=1}^{n} \sum_{j=1}^{3} \zeta_{ij}{}^\beta h_j{}^{\beta\alpha} - P\rho_i{}^\alpha = 0 \qquad (i = 1, 2, 3;\ \alpha = 1, \ldots, m) \tag{10}$$

$$h_j{}^{\beta\alpha} = \int_V \psi_\beta \varphi_{\alpha,j} dv, \qquad \rho_i{}^\alpha = \int_S k_i \varphi_\alpha ds$$

for which the expansions (5) and (9) are assumed finite and containing m and n terms respectively. When $2n > m$, the set (10) admits a nonunique solution $\sigma(n, m)$ converging on the norm of H_σ to a stress field feasible from the point of view of statics. When $n = m$, this set becomes underdefined $3n$ times in the three-dimensional case, n times in the two-dimensional case, and is definable in the one-dimensional case. This complies with the order of nondefinability of the differential equations of equilibrium.

The sequence of static solutions of the problem corresponding to the saddle point (3) is obtained in the form of a sequence $P(n, m) = \sup P(\sigma)$ with the conditions (10) and

$$f\left(\sum_{\beta=1}^{n} \zeta_{ij}{}^\beta \psi_\beta\right) \leqslant 0 \tag{11}$$

With m fixed the sequence $P(n,m)$ increases monotonously. We prove this as follows. Suppose two expansions (9) exist, one containing l terms, the other ($n<l$) terms. The inequality (11) defines in H_σ convex closed regions Ω^n and Ω^l situated in the linear envelopes embracing $\{\psi_1, \ldots, \psi_n\}$ and $\{\psi_1, \ldots, \psi_n, \ldots, \psi_l\}$. The convexity and the inclusion of the first basis in the second imply that Ω^n is contained in Ω^l ($\Omega^n \subset \Omega^l$). Since the number of equations in (10) is fixed, it follows that $\sigma(n, m) \subset \sigma(l, m)$. Under these conditions the inequality $P(n, m) > P(l, m)$ would violate the static limit equilibrium theorem. Consequently the sequence $P(n, m)$ does not diminish with increasing n and by virtue of its boundedness has the following limit

$$P(m) = \lim P(n,m) \quad (n \to \infty) \tag{12}$$

Denoting now the linear envelope of the functions $\{\varphi_1, \ldots, \varphi_m\}$ by R_m we find that $P(m)$ represents the saddle point of the Lagrangian functional (3) provided that $\mathbf{u} \in R_m$. By duality, $P(m)$ also represents the saddle point of (4) provided again that $\mathbf{u} \in R_m$. But then R_m expands to H_u with increasing m, while the sequence $P(m)$ cannot increase and has a limit given by

$$P^* = \lim_{m\to\infty} P(m) = \lim_{m\to\infty} \lim_{n\to\infty} P(n, m) \tag{13}$$

The above limits are not interchangeable, if only for the condition $2n > m$ which makes solution of (10) possible. The convergence of $P(n, m)$ to $P(m)$ and of $P(m)$ to P^* implies the existence of the inequalities

$$|P^* - P(m)| \leqslant \varepsilon_m, \qquad |P(m) - P(n,m)| \leqslant \varepsilon_{nm}$$

which in turn imply that

$$|P^* - P(n, m)| = |P^* - P(m) + P(m) - P(n,m)| \leqslant \varepsilon_m + \varepsilon_{nm}$$

This indicates that the sequence $P(n, m)$ converges to P^* when n and m $(2n > m)$

increase simultaneously

$$P^* = \lim P(n, m) \qquad (n \to \infty, \ m \to \infty) \tag{14}$$

In this case the convergence is not monotonous and correponds to the method which could be called "mixed" in analogy with the static and kinematic methods for which the sequences $P(n,m)$ and $P(m)$ are monotonous.

The limit solution of the problem, often called the optimal plan, defines the saddle point $L^*(P^*, \sigma^*, \mathbf{u}^*, \lambda^*)$. A unique solution $L^* = P^*$ exists for this point, as well as for $\mathbf{u}^*$ and λ^* connected by the law governing the flow. In the plastic regions where $f(\sigma^*) = 0$ the distribution of stress is unique, in the rigid regions on the other hand in which $f(\sigma^*) < 0$, the stress distribution is determined with accuracy of up to the self-balanced field σ° satisfying the homogeneous conditions $\sigma_{ij}{}^\circ \nu_j = 0$ on S_σ. The condition of plasticity $f(\sigma^* + \sigma^\circ) \leqslant 0$ must hold for σ° in the rigid region, and a converging sequence for the stresses can therefore be constructed by solving an incorrect programing problem after Tikhonov [3]. This means that the constraints (10) and (11) will be imposed not on $P(\sigma)$, but on the regularized functional

$$M_\alpha (P, \sigma^n) = P - \alpha_n \omega (\sigma^n) \tag{15}$$

representing the required function whose maximum is being sought. Here $\alpha_n > 0$ and $\alpha_n \to 0$ as $n \to \infty$, while $\omega(\sigma^n)$ is a stabilizing functional such as e.g. a suitably designated norm $\|\sigma\|$. In the present problem $\|\sigma\|$ can be determined in the S. L. Sobolev space $W_2{}^2$. The sequence σ^{n*} will then converge uniformly on the intervals of continuous differentiability provided that

$$M_\alpha [P(n, m), \sigma^{n*}] = \sup (P - \alpha_n \|\sigma^n\|^2)$$

For an incompressible material the norm of the stress tensor can be found in terms of the deviator s_{ij}

$$\|\sigma\|^2 = \int\limits_V (s_{ij}s_{ij} + \gamma s_{ij,kl}s_{ij,kl})\, dv \qquad \left(s_{ij} = \sigma_{ij} - \frac{\sigma_{ii}}{3}\right)$$

where γ is some positive function used to coordinate the dimensionality of the integrand. The quantity σ^{n*} and its first order derivatives converge uniformly on the segments of continuous differentiability to some optimal σ^* and the second order derivatives converge in the mean. All the above processes are accurate to within the hydrostatic pressure term.

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